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Q-Deformed Oscillator Algebra and an Index Theorem for the Photon Phase Operator

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Abstract

The quantum deformation of the oscillator algebra and its implications on the phase operator are studied from a view point of an index theorem by using an explicit matrix representation. For a positive deformation parameter q or $q = exp(2\pi i\theta)$ with an irrational θ, one obtains an index condition $dim\ \ ker\ \ a - dim\ \ ker\ \ a^{\dagger} = 1$ which allows only a non-hermitian phase operator with $dim\ \ ker\ \ \mathrm{e}^{i\varphi} - dim\ \ ker\ \ (\mathrm{e}^{i\varphi})^{\dagger} = 1$. For $q = exp(2\pi i\theta)$ with a rational θ , one formally obtains the singular situation $dim\ \ ker\ \ a = \infty$ and $dim\ \ ker\ \ a^{\dagger} = \infty$, which allows a hermitian phase operator with $dim\ \ ker\ \ \mathrm{e}^{i\Phi} - dim\ \ ker\ \ (\mathrm{e}^{i\Phi})^{\dagger} = 0$ as well as the non-hermitian one with $dim\ \ ker\ \ \mathrm{e}^{i\varphi} - dim\ \ ker\ \ (\mathrm{e}^{i\varphi})^{\dagger} = 1$. Implications of this interpretation of the quantum deformation are discussed. We also show how to overcome the problem of negative norm for $q = exp(2\pi i\theta)$.

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1 Introduction

The presence or absence of a hermitian phase operator for the photon is an old and interesting problem [1, 2, 3], see ref [4] for earlier works on the subject. Recently, one of the present authors [5] introduced the notion of index into the analysis of the phase operator. The basic observation is that the creation and annihilation operators of the oscillator algebra

$$[a, a^{\dagger}] = 1 \tag{1}$$

satisfy the index condition:[§]

$$dim \quad ker \quad a - dim \quad ker \quad a^{\dagger} = 1 \tag{2}$$

as seen from the conventional representation

$$a = |0><1| + |1><2|\sqrt{2} + |2><3|\sqrt{3} + \cdots . \tag{3}$$

The state vectors $|k>$ are defined by $N|k>= k|k>$, where N is the number operator. The phase operator defined by [2]

$$\begin{aligned} \mathrm{e}^{i\varphi} &= \frac{1}{\sqrt{N+1}} a \\ &= |0><1| + |1><2| + |2><3| + \cdots \end{aligned} \tag{4}$$

faithfully reflects the index relation (2)

$$dim \quad ker \quad \mathrm{e}^{i\varphi} - dim \quad ker \quad (\mathrm{e}^{i\varphi})^{\dagger} = 1. \tag{5}$$

On the other hand, if one assumes a polar decomposition $a = U(\phi)H$ with a unitary $U(\phi)$ and a hermitian H, one inevitably has

[§]The index of a linear operator a, for example, is defined as the number of normalizable states u_n which satisfy $au_n = 0$.

$$dim \quad ker \quad a - dim \quad ker \quad a^\dagger = 0 \tag{6}$$

since the action of the unitary operator $U(\phi)$ is simply to re-label the names of the basis vectors. From these considerations, one concludes that the phase operator φ in (4) cannot be hermitian, i.e., $e^{i\varphi}$ is not unitary. A truncation of the representation space of a to $(s+1) \times (s+1)$ dimensions, however, generally leads to the index relation (6)[5], and thus an associated phase operator ϕ could be hermitian. In fact, a hermitian phase operator ϕ may be defined by [3]

$$\mathrm{e}^{i\phi} = |0 >< 1| + |1 >< 2| + |2 >< 3| + \cdots + |s-1 >< s| + \mathrm{e}^{i\phi_0}|s >< 0| \tag{7}$$

with a positive integer s (a cut-off parameter) and an arbitrary constant ϕ_0. The unitary operator $\mathrm{e}^{i\phi}$ naturally satisfies the index condition

$$dim \quad ker \quad \mathrm{e}^{i\phi} - dim \quad ker \quad (\mathrm{e}^{i\phi})^\dagger = 0, \tag{8}$$

and gives rise to a truncated operator

$$\begin{aligned} a_s &= \mathrm{e}^{i\phi}\sqrt{N} \\ &= |0 >< 1| + |1 >< 2|\sqrt{2} + |2 >< 3|\sqrt{3} + \cdots |s-1 >< s|\sqrt{s} \end{aligned} \tag{9}$$

with

$$dim \quad ker \quad a_s - dim \quad ker \quad a_s^\dagger = 0 \tag{10}$$

since $\quad a_s^\dagger|s >= 0$.

The index relations (5) and (8) clearly show the unitary inequivalence of $\mathrm{e}^{i\varphi}$ and $\mathrm{e}^{i\phi}$ even for arbitrarily large s. Since the kernel of $a_s^\dagger$ is given by $ker\ a_s^\dagger = \{|s\rangle\}$ in (10), which is ill-defined in the limit $s \to \infty$, we analyze the behavior of $e^{i\phi}$ for sufficiently large but finite s. To make this statement of large s meaningful, we need to introduce a typical number to

characterize a physical system, relative to which the number s may be chosen much larger. We thus expand a physical state as

$$|p>=\sum_{n=0}^{\infty} p_n|n> . \tag{11}$$

The finiteness of $< p|N^2|p >$ requires

$$\sum_n n^2 \ |p_n|^2 = N_p^2 < \infty \tag{12}$$

in addition to the usual condition of a vector in a Hilbert space,

$$\sum_n |p_n|^2 < \infty. \tag{13}$$

The number N_p in (12) specifies a typical number associated to a given physical system $|p\rangle$. By choosing the parameter s at $s >> N_p$, one may analyze the physical implications of the state $|s\rangle$, which is responsible for the index in (10), on the physically observable processes. It was shown in [5] that the origin of the index mismatch between (4) and (7), namely the state $|s\rangle$ in (7), is also responsible for the absence of minimum uncertainty states for the hermitian operator ϕ in the characteristically quantum domain with small average photon numbers.

A major advantage of the notion of index is that it is invariant under unitary time developments which include a fundamental phenomenon such as squeezing. Another advantage of the index idea lies in suggesting a close analogy between the problem of quantum phase operator with a non-trivial index as in (5) and chiral anomaly in gauge theory, which is related to the Atiyah-Singer index theorem. This was emphasized in Ref [5]. From an anomaly view point, it is not surprising to have an anomalous identity

$$C(\varphi)^2 + S(\varphi)^2 = 1 - \frac{1}{2}|0 >< 0| \tag{14}$$

and an anomalous commutator

$$[C(\varphi), S(\varphi)] = \frac{1}{2i}|0><0| \tag{15}$$

for the modified cosine and sine operators defined in terms of $\mathrm{e}^{i\varphi}$ in (4) [2]

$$\begin{aligned} C(\varphi) &\equiv \frac{1}{2}\{\mathrm{e}^{i\varphi} + (\mathrm{e}^{i\varphi})^\dagger\}, \\ S(\varphi) &\equiv \frac{1}{2i}\{\mathrm{e}^{i\varphi} - (\mathrm{e}^{i\varphi})^\dagger\} \end{aligned} \tag{16}$$

The notion of index is also expected to be invariant under a continuous deformation such as the quantum deformation of the oscillator algebra as long as the norm of the Hilbert space is kept positive definite.

2 Q-deformation

The purpose of the present note is to analyze in detail the behavior of the index relation under the quantum deformation of the oscillator algebra [6, 7]:

$$\begin{aligned} [a, a^\dagger] &= [N+1] - [N] \\ [N, a^\dagger] &= a^\dagger \\ [N, a] &= -a \end{aligned} \tag{17}$$

where

$$[N] \equiv \frac{q^N - q^{-N}}{q - q^{-1}}. \tag{18}$$

The parameter q stands for the deformation parameter, and one recovers the conventional algebra in the limit $q \to 1$. The quantum deformation (17) is known to satisfy the Hopf structure [8, 9]. The algebra(17) accomodates a Casimir operator defined by [9]

$$c = a^\dagger a - [N] \tag{19}$$

which plays an important role in the following.

For a real positive q, we may adopt the conventional Fock state representation of the algebra (17) defined by [6, 7]:

$$
\begin{aligned}
c|0> &= 0 \\
a|0> &= 0 \\
<0|0> &= 1 \\
N|k> &= k|k> \\
|k> &= \frac{1}{\sqrt{[k]!}}(a^\dagger)^k|0> \\
a|k> &= \sqrt{[k]}|k-1>, \quad a^\dagger|k>= \sqrt{[k+1]}|k+1>,
\end{aligned}
\tag{20}
$$

Here we have abbreviated $|k>_q$ by $|k>$. For a positive q, one thus obtains a representation

$$
a = |0><1|\sqrt{[1]} + |1><2|\sqrt{[2]} + |2><3|\sqrt{[3]} + \cdots \tag{21}
$$

which satisfies the index condition (2). The phase operator $\mathrm{e}^{i\varphi}$ is defined by [10]

$$
\begin{aligned}
\mathrm{e}^{i\varphi} &= \frac{1}{\sqrt{[N+1]}}a \\
&= |0><1| + |1><2| + |2><3| + \cdots
\end{aligned}
\tag{22}
$$

so that the relation $a = \mathrm{e}^{i\varphi}\sqrt{N}$ holds. Evidently, expression (22) has the same form as that of Susskind and Glogower in [2], namely not only the index but also the explicit form of $\mathrm{e}^{i\varphi}$ itself remains invariant under quantum deformation.

If one extends the range of the deformation parameter q to complex numbers, which is consistent only for $|q| = 1$, one finds more interesting possibility. For previous discussions of this case from a finite dimensional cyclic representation, see papers in [11].

For a complex $q = \exp(2\pi i\theta)$ with a real θ, we adopt the following explicit matrix representation [12] of the algebra (17)

$$
\begin{aligned}
a &= \sum_{k=1}^{\infty} \sqrt{[k-n_0]+[n_0]}|k-1><k| \\
a^\dagger &= \sum_{k=1}^{\infty} \sqrt{[k+1-n_0]+[n_0]}|k+1><k| \\
N &= \sum_{k=0}^{\infty}(k-n_0)|k><k| \\
c &= [n_0] \;=\; \frac{1}{|\sin 2\pi\theta|}
\end{aligned}
\tag{23}
$$

Here the ket states $|k>$ stand for column vectors

$$
|0> \;=\; \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}, \quad |1> \;=\; \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}, \quad |2> \;=\; \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ \vdots \end{pmatrix}, \ldots. \tag{24}
$$

and the bra states stand for row vectors. The representation (20) may also be included in this matrix representation by letting $n_0 = 0$ and $c = 0$. In eq(23) the Casimir operator c for the algebra (17) is chosen so that $a^\dagger a > 0$ and the absence of negative norm is ensured. We note that

$$
\begin{aligned}
[k-n_0] &= \frac{\sin 2\pi(k-n_0)\theta}{\sin 2\pi\theta} \\
&= -\frac{\cos(2\pi k\theta)}{|\sin 2\pi\theta|} \\
&\leq \frac{1}{|\sin 2\pi\theta|}
\end{aligned}
\tag{25}
$$

if one chooses n_0 as in (23),

$$[n_0] \ = \ \frac{\sin(2\pi n_0\theta)}{|\sin 2\pi\theta|} \ = \ \frac{1}{|\sin 2\pi\theta|} \tag{26}$$

The argument of the square root in (23) is thus non-negative. This means that we have managed to overcome the problem of negative norm for $q = exp(2\pi i\theta)$. For irrational θ

$$[k - n_0] + [n_0] = 0 \tag{27}$$

only if $k = 0$.

We thus have the kernels, ker $a = \{|0>\}$ and ker $a^\dagger$ = empty, and the index condition

$$dim \quad ker \quad a - dim \quad ker \quad a^\dagger = 1 \tag{28}$$

for a positive q or $q = \exp(2\pi i\theta)$ with an irrational θ: this index relation allows only the non-hermitian phase operator defined in (22), namely

$$\begin{aligned} \mathrm{e}^{i\varphi} \ &= \ \frac{1}{\sqrt{[N+1]+[n_0]}}a \\ &= \ |0><1| + |1><2| + |2><3| + \cdots \end{aligned} \tag{29}$$

This expression together with $[N+1]+[n_0] \neq 0$ shows that $\mathrm{e}^{i\varphi}$ and a carry the same index, namely a unit index.

We next examine the representation (23) for a rational θ. To be specific, we consider the case $q = \exp(\frac{2\pi i}{(s+1)})$, i.e., $\theta = \frac{1}{s+1}$ with a positive integer s greater than one. One then obtains

$$[s+1] \ = \ \frac{q^{s+1} - q^{-s-1}}{q - q^{-1}} \ = \ 0 \tag{30}$$

In this case, the representation (23) becomes

$$
\begin{aligned}
a &= \sqrt{[1-n_0]+[n_0]}|0><1| + \cdots + \sqrt{[s-n_0]+[n_0]}|s-1><s| \\
&\quad + \sqrt{[1-n_0]+[n_0]}|s+1><s+2| + \cdots + \sqrt{[s-n_0]+[n_0]}|2s><2s+1| \\
&\quad + \cdots \\
N &= (-n_0)|0><0| + (1-n_0)|1><1| + \cdots + (s-n_0)|s><s| \qquad (31) \\
&\quad + (s+1-n_0)|s+1><s+1| + \cdots + (2s+1-n_0)|2s+1><2s+1| \\
&\quad + \cdots \\
c &= [n_0] = \frac{\sin(\frac{2\pi n_0}{s+1})}{\sin(\frac{2\pi}{s+1})} = \frac{1}{\sin(\frac{2\pi}{s+1})}
\end{aligned}
$$

where $a^\dagger$ is given by the hermitian conjugate of a and one may choose $n_0 = \dfrac{s+1}{4}$.

One may look at the representation (31) from two different view points. One way is to regard it reducible into an infinite set of irreducible $(s+1)$- dimensional representation specified by the eigenvalue of the Casimir operator $c = [n_l]$ $(= [n_0])$ where

$$
\begin{aligned}
n_l &= n_0 - l(s+1) \\
&= \frac{1}{4}(s+1) - l(s+1) \qquad (32)
\end{aligned}
$$

with $l = 0, 1, 2, \cdots$. We note that $-n_l$ stands for the lowest eigenvalue of N. In this case, the basic Weyl block is given by

$$
\begin{aligned}
a_s &= \sqrt{[1-n_0]+[n_0]}|0><1| + \cdots + \sqrt{[s-n_0]+[n_0]}|s-1><s| \\
a_s^\dagger &= (a_s)^\dagger \\
N_s &= (-n_0)|0><0| + (1-n_0)|1><1| + \cdots + (s-n_0)|s><s| \\
c &= [n_0] = \frac{1}{\sin(\frac{2\pi}{s+1})} \qquad (33)
\end{aligned}
$$

and other sectors are obtained by using the Casimir operator $c = [n_l](= [n_0])$ with the lowest eigenvalue of N at $-n_l$, $l = 1, 2, \cdots$. This is the standard representation commonly adopted for the case $\theta = \frac{1}{(s+1)}$. This finite dimensional representation inevitably leads to the index condition[5]

$$dim \quad ker \quad a_s - dim \quad ker \quad a_s^\dagger = 0 \tag{34}$$

and one may introduce the phase operator of Pegg and Barnett in (7), which is unitary $\mathrm{e}^{i\phi}(\mathrm{e}^{i\phi})^\dagger = (\mathrm{e}^{i\phi})^\dagger \mathrm{e}^{i\phi} = 1$ in $(s+1)$-dimensional space. The large s-limit of this construction leads to the problematic aspects arising from index mismatch analysed in Ref[5]. Also, the large s-limit of (33) does not lead to the standard representation (20) with well-defined Casimir operator, since $n_0 = \frac{s+1}{4}$ in (33).

Another view of the representation (31), which is interesting from an index consideration, is to regard (31) as an infinite dimensional representation specified by the Casimir operator $c = [n_0]$ with $-n_0$ the lowest eigenvalue of N. We then have the kernels

$$\begin{aligned} ker\ a &= \{|0>, |s+1> |2s+2>, \cdots\} \\ ker\ a^\dagger &= \{|s>, |2s+1>, \cdots\} \end{aligned} \tag{35}$$

and

$$dim \quad ker \quad a = \infty, \quad dim \quad ker \quad a^\dagger = \infty \tag{36}$$

Consequently, (31) corresponds to a *singular* point of index theory where the notion of index becomes ill-defined: we have no constraint on the phase operator arising from an index consideration. In fact, one may accomodate either the non-unitary $\mathrm{e}^{i\varphi}$ in (4), which is normally associated with

$$dim \quad ker \quad a - dim \quad ker \quad a^\dagger = 1,$$

or a unitary $\mathrm{e}^{i\Phi}$ defined by

$$\begin{aligned} e^{i\Phi} &= |0><1| + |1><2| + |2><3| + \cdots + e^{i\phi_0}|s><0| \\ &+|s+1><s+2| + \cdots + e^{i\phi_1}|2s+1><s+1| \\ &+\cdots \end{aligned} \tag{37}$$

with ϕ_0, ϕ_1, $\cdots$, real constants; unitary $e^{i\Phi}$ is normally associated with

$$dim \quad ker \quad a - dim \quad ker \quad a^\dagger = 0.$$

Both of these phase operators give rise to the same representation for a in (31),

$$\begin{aligned} a &= e^{i\varphi}\sqrt{[N]+[n_0]} \\ &= e^{i\Phi}\sqrt{[N]+[n_0]} \end{aligned} \tag{38}$$

However, we have no more the expression in (29) since $[N+1]+[n_0]$ can vanish. The operator $e^{i\Phi}$ gives rise to the same physical implications as $e^{i\phi}$ in (7) for the physical states defined in (12).

3 Discussion and Conclusion

We would like to summarize the implications of the above analysis. First of all, the notion of index is well-defined for a real positive q (which includes $q = 1$), and the index is invariant under a continuous deformation specified by q. The notion of index presents a stringent constraint on the possible form of the phase operator.

For $q = \exp(2\pi i\theta)$, the notion of index becomes subtle. Since the rational values of θ are densely distributed among the real values of θ, one cannot define a notion of continuous deformation for the index (i.e., $dim \quad ker \quad a - dim \quad ker \quad a^\dagger$); one encounters singular points associated with a rational θ almost everywhere. Only when one regards the singular situation such as in (36) as corresponding to the index relation

$$dim \quad ker \quad a - dim \quad ker \quad a^{\dagger} = 1 \tag{39}$$

one maintains the notion of continuous deformation. Even in this case, there is certain complication for $\theta \to 0$ to reproduce the normal case of $q = 1$ if one sticks to representation (23); the Casimir operator cannot be well-defined in the limit $\theta \to 0$ as it should be.

If one formally defines the representation [§]

$$\begin{aligned} a &= \sum_{k=1}^{\infty} \sqrt{[k]} |k-1><k| \\ a^{\dagger} &= \sum_{k=1}^{\infty} \sqrt{[k+1]} |k+1><k| \\ N &= \sum_{k=0}^{\infty} k|k><k| \\ c &= 0 \end{aligned} \tag{40}$$

for all allowed values of q and if one formally takes the index (39) even for a rational θ, one can maintain the notion of continuous deformation of the algebra and its representation. Only in this case, the index as well as the phase operator remain invariant under q-deformation. The standard finite dimensional representation for $q = \exp(2\pi i\theta)$ with a rational θ may be interpreted that the well-defined notion of index, which is supposed to be invariant under deformation, is lost for a rational θ and the representation makes a discontinuous transition to finite dimensional irreducible representations.

In conclusion, the notion of index, when it is well-defined, is useful as an invariant characterization of q-deformation of an algebra. In addition, we have also shown how to overcome the problem of negative norm for $q = \exp(2\pi i\theta)$.

[§]Note that representation (40) generally contains negative norm states for $q = \exp(2\pi i\theta)$

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